MUMBAI UNIVERSITY

SEMESTER 2

APPLIED MATHEMATICS SOLVED PAPER – MAY 2018

N.B:- (1) Question no. 1 is compulsory.

(2) Attempt any 3 questions from remaining five questions.

Q.1.(a) **Evaluate** $\int_0^\infty 5^{-4x^2}\ dx$ **[3]**

Ans: **Let** I = $\int_0^\infty 5^{-4x^2}\ dx$

put $5^{-4x^2} = e^{-t}$

taking log on both sides,

$$4x^2 \log 5 = t$$

$$x^2 = \frac{t}{4\log 5} \quad => \quad x = \frac{\sqrt{t}}{2\sqrt{\log 5}}$$

diff. w.r.t x,

$$dx = \frac{t^{-1/2}}{4\sqrt{\log 5}} dt \qquad \text{lim}\longrightarrow[\,0, \infty\,]$$

$$\therefore I = \int_0^\infty \frac{e^{-t}}{4\sqrt{\log 5}}\ t^{-1/2}$$

$$\therefore I = \frac{1}{4\sqrt{\log 5}} \int_0^\infty e^{-t}.t^{-1/2}\ dt$$

$$\boxed{\therefore I = \frac{\sqrt{\pi}}{4\sqrt{\log 5}}} \qquad \ldots\ldots\ldots\{\ \int_0^\infty e^{-t}.t^{-1/2}\ dt = \sqrt{\pi}\ \}$$

(b) Solve $\frac{dy}{dx} = x.y$ $with$ **help of Euler's method ,given that y(0)=1 and find y when x=0.3** **[3]**

(Take h=0.1)

Ans : $\frac{dy}{dx} = x.y = f(x,y) \qquad x_0 = 0\,, y_0 = 1$

$$y_n = y_{n-1} + h.f(x_{n-1}, y_{n-1})$$

Iteration (n)	x_n	y_n	f(x_n, y_n)	$y_{n+1} = y_n + h.\mathrm{f}(x_n, y_n)$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302

$\therefore$y(0.3)=1.0302

(c) Evaluate $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y=0$ **[3]**

Ans: put $\frac{d}{dx}$ =D

$$\therefore\ D^4y + 2D^2y + y = 0$$

$$\therefore\ D^4 + 2D^2 + 1 = 0$$

Put $D^2 = t$

=> $t^2 + 2t + 1 = 0$

=> $t = -1, -1$

Roots are : D=+i,-i,+i,-i

The complementary solution of given eqn is

$$y_c = y_g = (C_1 + xC_2)\cos x + (C_3 + xC_4)\sin x$$

(d) Evaluate $\int_0^1 \sqrt{\sqrt{x} - x}\ dx$ **[3]**

Ans : Let I = $\int_0^1 \sqrt{\sqrt{x} - x}\, dx$

I = $\int_0^1 \sqrt{(\sqrt{x} - \sqrt{x}.\sqrt{x}}\ dx$

Take $\sqrt{x}\ common$,

I = $\int_0^1 x^{1/4} \sqrt{1 - x^{1/2}}$ dx

Put $x^{1/2}$= t

Squaring both sides,

$\therefore x = t^2$

Differentiate w.r.t x,

$\therefore dx = 2t.dt$

Limits after substitution : Lim ⟶[0,1]

$\therefore I = \int_0^1 t^{1/2}\sqrt{1-t}\ .\ 2.t\ dt$

$= 2\int_0^1 t^{3/2}\sqrt{1-t}\ dt$

$= 2\ \beta\left(\frac{5}{2},\frac{3}{2}\right)$ ......$\{\int_0^1 t^m.(1-t)^n = \beta(m+1,n+1)\}$

$\therefore I = \frac{\pi}{8}$

(e) Solve: $(1+\log x.y)dx + (1+\frac{x}{y})dy = 0$ [4]

Ans : Compare given eqn with Mdx+Ndy=0

$\therefore M = (1+\log x.y)$ $\therefore N = 1+\frac{x}{y}$

$\frac{\partial M}{\partial y} = \frac{1}{xy}x = \frac{1}{y}$ $\frac{\partial N}{\partial x} = \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential eqn is exact.

The solution of exact differential eqn is given by,

$\int Mdx + \int\left(N - \frac{\partial}{\partial y}\int Mdx\right) dy = c$ ----------------(1)

$\int Mdx = \int(1+\log xy)\ dx = x + \log xy.x - x = x.\log xy$

$\frac{\partial}{\partial y}\int Mdx = \frac{x}{y}$

$\int\left(N - \frac{\partial}{\partial y}\int Mdx\right)$ dy = $\int\left(1+\frac{x}{y}-\frac{x}{y}\right) dy = y$

From eqn (1), the solution of given differential eqn is ,

x.log xy + y = c

(f) Evaluate I = $\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dx\cdot dy}{1+x^2+y^2}$ **[4]**

Ans : $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx.dy}{1+x^2+y^2}$

$I = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$

$\therefore I = \int_0^1 \frac{\pi}{4} \frac{1}{\sqrt{1+x^2}} \; dx$

$\therefore I = \frac{\pi}{4} \left[\log\left(x+\sqrt{1+x^2}\right)\right]_0^1$

$$\therefore I = \frac{\pi}{4} \log(1+\sqrt{2})$$

Q.2. (a) Solve $xy(1+xy^2)\frac{dy}{dx} = 1$ [6]

Ans: $\therefore \frac{dx}{dy} = xy + x^2y^3$

$\therefore \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x}y = y^3$ Now, put $-\frac{1}{x} = v$

$\therefore \frac{dv}{dy} + vy = y^3$ ....$\left(\frac{1}{x^2}\frac{dx}{dy} = \frac{dv}{dy}\right)$

This is linear differential eqn.

$\therefore$ Integrating Factor$=e^{\int y dy} = e^{\frac{y^2}{2}}$

The solution of linear diff. eqn is given by,

$v.(\text{I.F.}) = \int (I.F.)(R.H.S) + c$

$v\, e^{\frac{y^2}{2}} = \int e^{\frac{y^2}{2}}.y^3 \, dy + c$

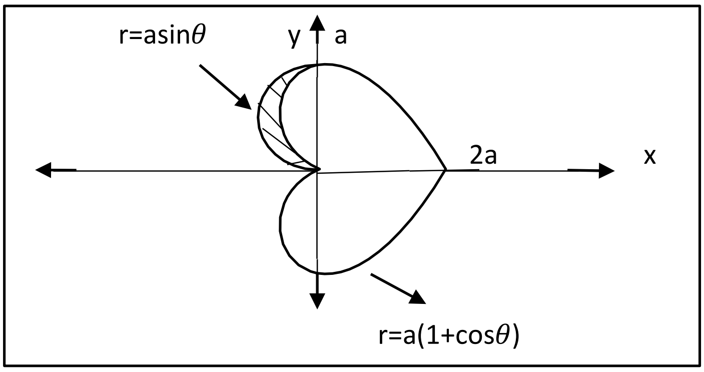
$$-\frac{1}{x} e^{\frac{y^2}{2}} = e^{\frac{y^2}{2}}(y^2-2) + c$$

Where c is constant of integration.

(b) Find the area inside the circle r=a sinθ and outside the cardioide r=a(1+cosθ) [6]

Ans : Intersection of cardioide and circle is,

r=a(1+cosθ) and r=asinθ



$$a\sin\theta = a(1+\cos\theta) \quad => \quad \theta = 90^\circ$$

$$a(1+\cos\theta) \le r \le a\sin\theta$$

$$\frac{\pi}{2} \le \theta \le \pi$$

Area of region bounded by given circle and cardioide ,

$$I = \int_{\frac{\pi}{2}}^{\pi} \int_{a\sin\theta}^{a(1+\cos\theta)} r\, dr d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{a^2}{2}(\sin^2\theta - 1 - 2\cos\theta - \cos^2\theta)\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \frac{a^2}{2}(-1 - 2\cos\theta - \cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[-\theta - 2\sin\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$$

$$I = \frac{a^2}{2}\left[(-\pi - 0 - 0) - \left(-\frac{\pi}{2} - 2 - 0\right)\right]$$

Required area is = $I = \frac{a^2}{2}\left(2 - \frac{\pi}{2}\right)$

(c) Apply Rungee-Kutta Method of fourth order to find an approximate value of y when x=0.2 given that $\frac{dy}{dx} = x + y$ when y=1 at x=0 with step size h=0.2. [8]

Ans: $\frac{dy}{dx} = x + y \qquad x_0 = 0 \;,\; y_0 = 1$, h= 0.2

f(x,y)= x + y

k_1= h.f(x_0, y_0) = 0.2 f(0,1) = 0.2

k_2 = h.f($x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}$) = 0.2 f(0.1,1.1) = 0.24

$k_3 = h.f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2f(0.1, 1.12) = 0.244$

$k_4 = h.f(x_0 + h, y_0 + k_3) = 0.2\ f(0.2, 1.244) = 0.2888$

$k = \frac{k_1+2k_2+2k_3+k_4}{6} = \frac{0.24+0.48+0.488+0.2888}{6} = 0.2428$

The value of y at x=0.2 is given by,

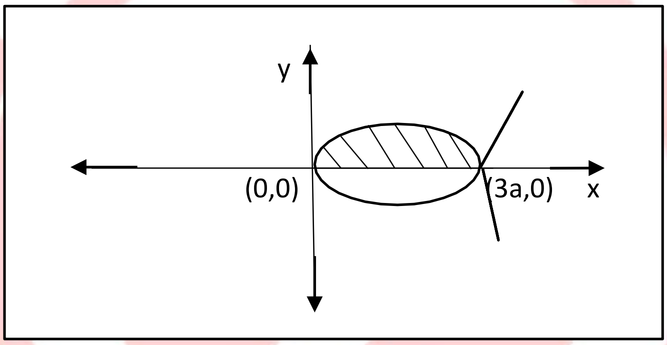
$$y(0.2) = y_0 + k = 1 + 0.2428$$

$$\boxed{y(0.2) = 1.2428}$$

Q.3 (a) Show that the length of curve $9ay^2 = x(x-3a)^2$ is $4\sqrt{3}a$. [6]

Ans : Curve : $9ay^2 = x(x-3a)^2$ .... (1)

The given curve is strophoid.



Differentiate eqn (1) w.r.t x,

$$18ay\frac{dy}{dx} = 2x(x-3a) + (x-3a)^2$$

$$\therefore 18ay\frac{dy}{dx} = 3(x-3a)(x-a)$$

$$\therefore \frac{dy}{dx} = \frac{(x-3a)(x-a)}{6ay}$$

Squaring both the sides,

$$(\frac{dy}{dx})^2 = \frac{(x-3a)^2(x-a)^2}{36\,a^2y^2}$$

$$\therefore (\frac{dy}{dx})^2 = \frac{(x-3a)^2(x-a)^2}{4a\,x\,(x-3a)^2} \quad \text{.................. from (1)}$$

$$\therefore (\frac{dy}{dx})^2 = \frac{(x-a)^2}{4ax}$$

The perimeter of given curve is ,

$$S = \int_0^{3a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{3a} \sqrt{1 + \frac{(x-a)^2}{4ax}}\, dx = \int_0^{3a} \sqrt{\frac{(x+a)^2}{4ax}}\, dx$$

$$\therefore S = \int_0^{3a} \frac{x + a}{2\sqrt{x}\sqrt{a}}\, dx$$

$$\therefore S = \frac{1}{2\sqrt{a}} \int_0^{3a} \frac{x+a}{\sqrt{x}}\, dx$$

$$= \frac{1}{2\sqrt{a}} \left[\frac{2x\sqrt{x}}{3} + 2\sqrt{x} \right]_0^{3a}$$

$$= \frac{1}{2\sqrt{a}} \left(\frac{2a\sqrt{3a}}{1} + 2\sqrt{3a} \right)$$

$\therefore S = 2\sqrt{3}$ ---------------- (Half curve length)

$\therefore$ The total length of given curve = 2 S = 4 $\sqrt{3}$ units.

(b) Change the order of integration of $\int_0^1 \int_{-\sqrt{2y-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy.$ **[6]**

Ans : Let $I = \int_0^1 \int_{-\sqrt{2y-y^2}}^{1+\sqrt{1-y^2}} f(x,y)\,dx\,dy$

Region of integration : $-\sqrt{2y-y^2} \le x \le 1 + \sqrt{1-y^2}$

$$0 \le y \le 1$$

Curves : (i) $x = -\sqrt{2y-y^2}$ => $x^2 + y^2 = 2y$ => $x^2 + (y-1)^2 = 1$

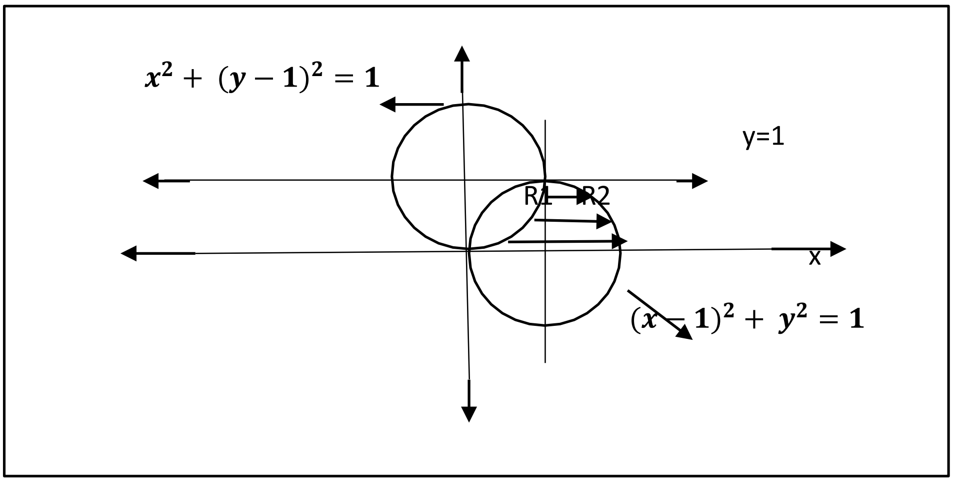
Circle with centre (0,1) and radius 1.

(ii) $x = 1 + \sqrt{1-y^2}$ => $(x-1)^2 + y^2 = 1$

Circle with centre (1,0) and radius 1.

(iii) $y = 0$ *line i.e equation of* $x -$ *axis*.

(iv) $y = 1$ *line parallel to* $x -$ *axis*.



Divide the region R into R1 and R2

$\therefore$ R = R1 ∪ R2

After changing the order of integration ,

For region R1 : $0 \le y \le 1 - \sqrt{1 - x^2}$

$0 \le x \le 1$

For region R2 : $0 \le y \le \sqrt{1 - (x-1)^2}$

$1 \le x \le 2$

As the region is divided in two parts the integration will be the union of the two region limits.

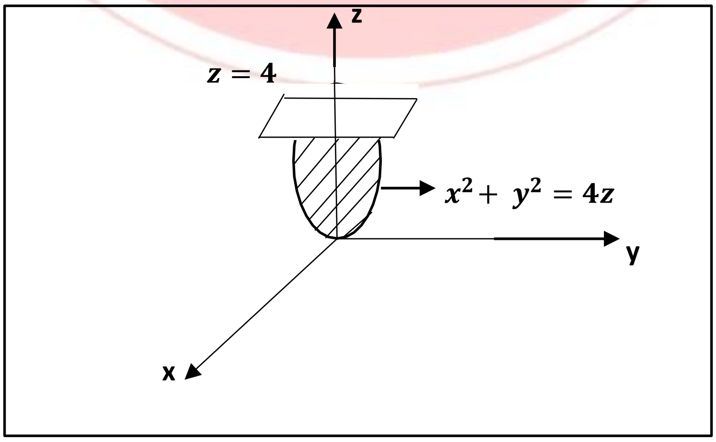
$$I = \int_0^1 \int_0^{1-\sqrt{1-x^2}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{\sqrt{1-(x-1)^2}} f(x,y)\,dy\,dx$$

This is the integration after changing order from dx dy to dy dx of given integration region.

(c) Find the volume of the paraboloid $x^2 + y^2 = 4z$ cut off by the plane $z = 4$

[8]

Ans: Paraboloid : $x^2 + y^2 = 4z$ Plane : $z = 4$



Cartesian coordinate ⟶ cylindrical coordinates

$$(x, y, z) \longrightarrow (r, \theta, z)$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ $\quad \therefore x^2 + y^2 = r^2$

$\therefore$ Paraboloid : $r^2 = 4z$ and Plane : z = 4

If we are passing one arrow parallel to z axis from –ve to +ve we will get limits of z

$$\therefore \quad \frac{r^2}{4} \le z \le 4$$

$$0 \le r \le 4$$

$$0 \le \theta \le \frac{\pi}{2}$$

Volume of given paraboloid cut off by the plane is given by ,

$$V = 4 \int_0^{\frac{\pi}{2}} \int_0^4 \int_{\frac{r^2}{4}}^4 r \, dr \, d\theta \, dz$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^4 [\, 4r - \frac{r^4}{16} \,]_{\frac{r^2}{4}}^{4} \, dr \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} \int_0^4 (\, 4r - \frac{r^3}{4}) \, dr \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} [\, 2r^2 - \frac{r^4}{16} \,]_0^4 \, d\theta$$

$$= 4 \int_0^{\frac{\pi}{2}} (32 - 16) \, d\theta$$

$$V = 32\pi \text{ cubic units}$$

Q.4 (a) Show that $\int_0^1 \frac{x^a - 1}{\log x}$ dx = log (a+1) **[6]**

Ans : let I = $\int_0^1 \frac{x^a - 1}{\log x}$ dx

Taking 'a' as parameter ,

$$I(a) = \int_0^1 \frac{x^a - 1}{\log x} \, dx \qquad \text{-------- (1)}$$

differentiate w.r.t a ,

$$\frac{dI(a)}{da} = \frac{d}{da} \int_0^1 \frac{x^a - 1}{\log x} \, dx$$

$$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{\partial}{\partial a} \frac{x^a - 1}{\log x} \, dx \qquad \text{.........\{ D.U.I.S f(x)\}}$$

$\therefore \frac{dI(a)}{da} = \int_0^1 \frac{x^a . \log x}{\log x}$ dx ………$\{ \frac{dx^a}{da} = x^a . \log a \}$

$\therefore \frac{dI(a)}{da} = \int_0^1 x^a \ dx$

$\therefore \frac{dI(a)}{da} = [\frac{x^{a+1}}{a+1}]_0^1$

$\therefore \frac{dI(a)}{da} = \frac{1}{a+1} - 0$

$\therefore \frac{dI(a)}{da} = \frac{1}{a+1}$

now , integrate w.r.t a,

$I(a) = \int \frac{1}{a+1} \ da$

I(a) = log (a+1) + c -------- (2)

where c is constant of integration

put a=0 in eqn (1),

$I(0) = \int_0^1 0 \ dx = 0$

And

From eqn (2), I(0)= c

$\therefore$ c = 0

$\therefore$ I = log(a+1) **Hence proved.**

(b) If y satisfies the equation $\frac{dy}{dx} = x^2 y - 1$ $with$ $x_0 = 0, y_0 = 1$ using Taylor's Series Method find y at $x = 0.1$ (take h=0.1). **[6]**

Ans : $\frac{dy}{dx} = x^2 y - 1$ $x_0 = 0, y_0 = 1 , h = 0.1$

To find : $y(0.1)$

$y' = x^2 y - 1$, $y'_0 = -1$

$y'' = x^2 y' + 2xy$, $y''_0 = 0$

$y''' = x^2 y'' + 2y' x + 2y + 2x y'$, $y'''_0 = 2$

Taylor's series is :

$$y = y_0 + h.y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

$$\therefore y(0.1) = 1 + 0.1(-1) + 0 + \frac{(0.1)^3}{3!} (2)$$

$$\therefore y(0.1) = 0.9003$$

(c) Find the value of the integral $\int_0^1 \frac{x^2}{1+x^3} dx$ using (i)Trapezoidal rule (ii) Simpson's $(1/3)^{rd}$ rule (iii)Simpson's $(3/8)^{th}$ rule. [8]

Ans : Let I = $\int_0^1 \frac{x^2}{1+x^3}$ dx

a=0 , b=1

Dividing limits into 4 parts i.e n=4 $\therefore$ h = $\frac{b-a}{n}$ = $\frac{1}{4} = 0.25$

x_0 =0	x_1 =0.25	x_2 =0.50	x_3 =0.75	x_4 =1.0
y_0 =0	y_1 =0.06153	y_2 =0.2222	y_3 =0.39560	$y_4 = 0.5$

(i) Trapezoidal rule : I = $\frac{h}{2}[X + 2R]$ ------------------(1)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_4 = 0 + 0.5 = 0.5$$

$$R = sum\ of\ remaining\ ordinates = y_1 + y_2 + y_3$$
$$= 0.06153 + 0.2222 + 0.39560 = 0.67933$$

I = $\frac{0.25}{2}(0.5 + 2(0.39560))$ ....(from 1)

$\therefore$ I = 0.2323

(ii) Simpson's $(1/3)^{rd}$ rule :

I = $\frac{h}{3}[X + 2E + 4O]$ ----------------(2)

$$X = sum\ of\ extreme\ ordinates = y_0 + y_4 = 0 + 0.5 = 0.5$$

$$E = sum\ of\ even\ base\ ordinates = y_2 = 0.2222$$

$$O = sum\ of\ odd\ base\ ordinates = y_1 + y_3 = 0.06153 + 0.39560 = 0.45713$$

I = $\frac{0.25}{3}(0.5 + 2 \times 0.2222 + 4 \times 0.45713)$ ....(from 2)

$\therefore$ I = 0.23108

(iii) Simpson's $(3/8)^{th}$ rule :

$$I = \frac{3h}{8}[\,X + 2T + 3R\,] \qquad \text{-------------(3)}$$

$$X = sum\ of\ extreme\ ordinates = y_0 + y_4 = 0 + 0.5 = 0.5$$

$$T = sum\ of\ multiple\ of\ three\ base\ ordinates = y_3 = 0.39560$$

$$R = sum\ of\ remaining\ ordinates = y_1 + y_2 = 0.06153 + 0.2222 = 0.28373$$

$$I = \frac{3 \times 0.25}{8}(0.5 + 2 \times 0.39560 + 3 \times 0.28373)$$

$$\therefore I = 0.2008$$

Q. 5 (a). Solve $(y - xy^2)dx - (x + x^2y)dy = 0$ **[6]**

Ans : $(y - xy^2)dx - (x + x^2y)dy = 0$ -----------------------(1)

Comparing the given eqn with M dx +N dy = 0

$\therefore M = (y - xy^2)$ $\qquad \therefore N = -(x + x^2y)$

$\frac{\partial M}{\partial y} = 1 - 2xy$ $\qquad \frac{\partial N}{\partial x} = -(1 + 2xy)$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given differential eqn is not exact diff. eqn.

But the given diff . eqn is in the form of $y.f(xy)dx + xf(xy)dy = 0$

Integrating factor = I.F. = $\frac{1}{Mx - Ny} = \frac{1}{xy - x^2y^2 + xy + x^2y^2} = \frac{1}{2xy}$

Multiply the I.F. to eqn (1)

$$\left(\frac{1}{2x} - \frac{y}{2}\right) dx - \left(\frac{1}{2y} + \frac{x}{2}\right) dy = 0$$

$\therefore M_1 = \left(\frac{1}{2x} - \frac{y}{2}\right)$ $\qquad N_1 = -\left(\frac{1}{2y} + \frac{x}{2}\right)$

$$\int M_1 dx = \int \left(\frac{1}{2x} - \frac{y}{2}\right) dx = \frac{1}{2}(\log x) - \frac{xy}{2}$$

$$\frac{\partial}{\partial y}\int M_1\, dx = \frac{x}{-2}$$

$$\int [\, N_1 - \frac{\partial}{\partial y}\int M_1\, dx\,]dy = \int \frac{-1}{2y}\, dy = \frac{-1}{2}(\log y)$$

The solution of given diff. eqn is given by,

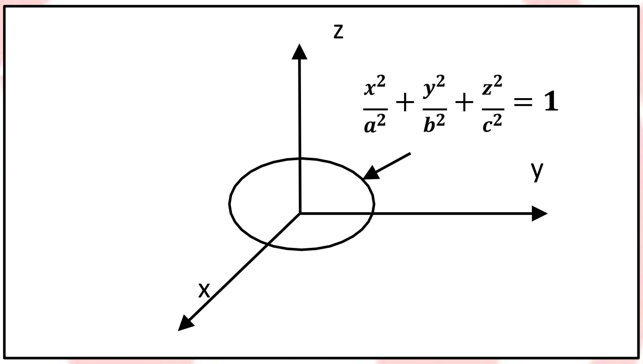
$$\int M_1 dx + \int [\, N_1 - \frac{\partial}{\partial y} \int M_1\, dx \,] dy = c$$

$$\therefore\ \frac{1}{2}(\log x) - \frac{xy}{2} - \frac{1}{2}\,(\log y) = c$$

$$\therefore\ \log\left(\frac{\sqrt{x}}{\sqrt{y}}\right) - \frac{xy}{2} = c$$

(b) Evaluate $\int\int\int \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\ dx\, dy\, dz$ **over the ellipsoid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. **[8]**

Ans : **Ellipsoid :** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$



Cartesian coordinates ⟶ **spherical coordinate system**

(x, y, z) ⟶ $(r, \theta, \emptyset)$

Put $x = a\, r \sin\theta \cos\emptyset$, $y = b\, r \sin\theta \sin\emptyset$, $z = c\, r\cos\theta$

$$dx\, dy\, dz = abc\, r^2 \sin\theta\, dr\, d\theta\, d\emptyset$$

$$\therefore\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$$

$$f(x, y, z) = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}} = \sqrt{1 - r^2} = f(r, \theta, \emptyset)$$

Limits : $0 \le r \le 1$

$$0 \le \theta \le \frac{\pi}{2}$$

$$0 \le \emptyset \le \frac{\pi}{2}$$

$\therefore$ I = $8 \int\int\int \sqrt{1-r^2}\ abc\, r^2 \sin\theta\, dr\, d\theta\, d\emptyset$

= $8 \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{r} \sqrt{1-r^2}\ abc\, r^2 \sin\theta\, dr\, d\theta\, d\emptyset$

= $8\, abc \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta \int_0^{\frac{\pi}{2}} d\emptyset \int_0^{r} \sqrt{1-r^2}\ r^2\, dr$

= $8\, abc\ [-\cos\theta]_0^{\frac{\pi}{2}}\ [\emptyset]_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \cos t . \sin^2 t . \cos t\, dt$ ---------{ put r = sint}

= $8\, abc\ (\frac{\pi}{2})(\frac{\pi}{8})$ …………{ usi$\beta\ f^n$ }

$\therefore$ I = $\frac{\pi^2}{4}\ (abc)$

(c) Evaluate $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$ **[8]**

Ans : $(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$ …………(1)

Put $(2x+1) = e^z$ => $x = \frac{e^z-1}{2}$

$\frac{dz}{dx} = \frac{2}{(2x+1)}$ but $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2\frac{dy}{dz} = \frac{2}{(2x+1)}$ D y where D=$\frac{d}{dz}$

$\therefore$ (2x+1)$\frac{dy}{dx} = 2\, Dy$

$\therefore (2x+1)^2 \frac{d_2y}{d^{x_2}} = 2^2\, D\,(D-1)y$

From (1),

$4D(D-1)y - 4\,Dy - 12y = 6(\frac{e^z-1}{2})$

$(4\,D^2 - 8D - 12)y = 3(e^z - 1)$

For complementary solution ,

$(4\,D^2 - 8D - 12) = 0$

$\therefore\ D = -1, 3$

$\therefore\ y_c = c_1 e^{-z} + c_2 e^{3z}$

For particular integral ,

$y_p = \frac{1}{f(D)}\, X$

$$y_p = \frac{1}{4D^2-8D-12} \ (3(e^z-1))$$

$$\therefore \ y_p = \frac{3}{4}\frac{1}{D^2-2D-3} \ (e^z-1)$$ put D = a = 1 and D = a = 0

$$\therefore \ y_p = \frac{3}{4}\ (\frac{1}{3} - \frac{e^z}{4})$$

The general solution of given differential eqn is ,

$$\therefore \ y_g = y_c + \ y_p = \ c_1 e^{-z} + c_2 e^{3z} + \frac{3}{4}\ (\frac{1}{3} - \frac{e^z}{4})$$

Resubstituting z **,**

$$\therefore \ y_g = \ c_1(2x+1)^{-1} + \ c_2(2x+1)^3 + \frac{3}{4}\ (\frac{1}{3} - \frac{(2x+1)}{4})$$

Q.6.(a) A resistance of 100 ohms and inductance of 0.5 henries are connected in series With a battery of 20 volts. Find the current at any instant if the relation between L,R,E is L $\frac{di}{dt} + Ri = E.$ **[6]**

Ans : $$L\frac{di}{dt} + Ri = E$$

$$\therefore \quad \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

Solution is given by ,

$$i.e^{\int(\frac{R}{L})_d^{\ t}} = \int e^{\int(\frac{R}{L})_{dt}}.\frac{E}{L}\,dt + c$$

$$\therefore \ \ i.e^{(Rt/L)} = \frac{E\,e^{(Rt/L)}}{R} + c$$

At t=0, i=0 $\therefore$ c = -$\frac{E}{R}$

$$\therefore \ i.e^{(Rt/L)} = \frac{E\,e^{(Rt/L)}}{L} + \frac{-E}{R}$$

$$\therefore i = \frac{E}{R}\ (1 - \ e^{-(Rt/L)})$$

For given condition R = 100, L = 0.5, E = 20

$$\therefore i = 0.2(1 - e^{-200t})$$

(b) Solve by variation of parameter method $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$.

[6]

Ans : $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$$

Put D = $\frac{d}{dx}$ $\quad \therefore D^2y + 3Dy + 2y = 0$

For complementary solution,

f(D)=0

$$\therefore D^2 + 3D + 2 = 0$$

$$D = -1, -2$$

$$\boxed{\therefore \; y_c = c_1e^{-x} + c_2e^{-2x}}$$

Particular integral is given by ,

$$y_p = y_1p_1 + y_2p_2$$

where $p_1 = \int \frac{-y_2 X}{w} dx$

$$p_2 = \int \frac{y_1 X}{w} dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$\therefore w = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -e^{-3x}$$

$p_1 = \int \frac{e^{-2x}.e^{e^x}}{e^{-3x}} dx = \int e^{e^x}.e^x dx = \int e^t \, dt = e^{e^x}$ **……{ put** $e^x = t \Rightarrow e^x dx = dt\}$

$$p_2 = \int \frac{e^{-x}}{-e^{-3x}}.e^{e^x} dx = \int e^{e^x}.e^{2x} dx = \int t.e^t dt = e^xe^{e^x} - e^{e^x}$$

$$\boxed{\therefore y_p = e^xe^{e^x} - (e^xe^{e^x} - e^{e^x}).e^{-2x} = e^{-2x}.e^{e^x}}$$

The general solution of given differential eqn is given by ,

$$\boxed{y_g = y_c + y_p = c_1e^{-x} + c_2e^{-2x} + e^{-2x}.e^{e^x}}$$

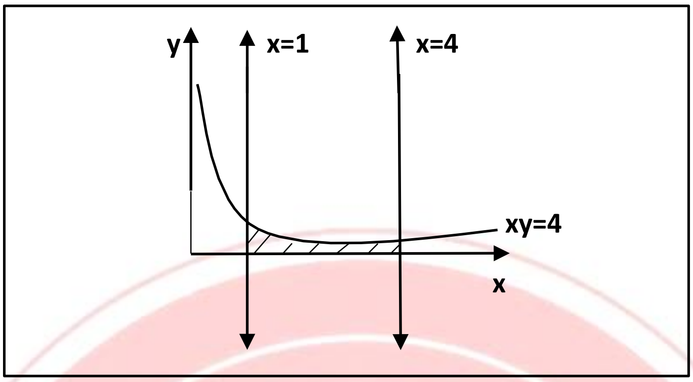
(c) Evaluate $\int\int xy(x-1)dx\,dy$ **over the region bounded by** $xy = 4, y = 0, x = 1$ **and** $x = 4$ **[8]**

Ans : Let I = $\int\int xy(x-1)dx\,dy$

Rectangular hyperbola : $xy = 4$ **Lines :** $x = 1, x = 4, y = 0$

Intersection of line $x = 1$ and $xy = 4$ is (1,4).

Intersection of line $x = 4$ and $xy = 4$ is (4,1)



$\therefore$ $$0 \le y \le \frac{x}{4}$$

$$1 \le x \le 4$$

$\therefore$ I = $\int_1^4 \int_0^{\frac{x}{4}} (x^2 y - xy) dy\, dx$

= $\int_1^4 \left[\frac{y^2}{2} x^2 - \frac{y^2 x}{2} \right]_0^{\frac{x}{4}} dx$

= $\int_1^4 (8 - \frac{8}{x})\ dx$

= $[\, 8x - 8 log\, x \ ]_1^4$

$\therefore$ **I = $8(3 - 2\, log\, 2)$**